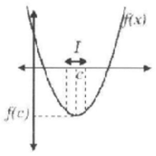
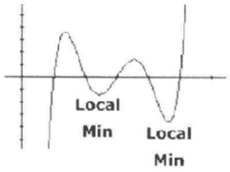
3.1
Extrema of functions

Definition of Local Minimum

$f(c)$ is the local minimum value of f

if $f(c) \le f(x)$ for every x in I.

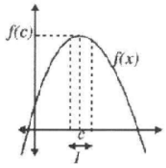


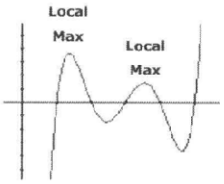


Definition of Local Maximum

$f(c)$ is the local maximum value of f

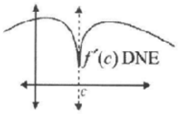
if $f(c) \ge f(x)$ for every x in I.

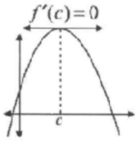




Theorem

If a function f has a local extremum at a number c in an open interval, then either $f'(c)=0$ or $f'(c)$ doesn't exist.





Critical numbers

A number c in the domain of a function f is a critical number of f if either $f'(c)=0$ or $f'(c)$ does not exist.

Example 1

If $f(x)=4x^3$, prove that f has no local extrema.

roof Since a local extremum must occur at a critical number, **find the critical numbers.**

Take the derivative: $f'(x)=4\cdot 3x^2$

$f'(x)=0$ $f'(x)$ ~~DNE~~

$12x^2=0$

$x=0$

If $x<0 \rightarrow f(x)<0$, if $x>0 \rightarrow f(x)>0$

So, $x=0$ is not the lowest or the highest point on the small interval around it.

The function has no local extrema.

Example 2

Find the critical numbers of $f(x)=\dfrac{2x-3}{x^2-1}$

$$f'(x)=\frac{(2x-3)'(x^2-1)-(x^2-1)'(2x-3)}{(x^2-1)^2}=\frac{2(x^2-1)-2x(2x-3)}{(x^2-1)^2}$$

$$=\frac{2x^2-2-4x^2+6x}{(x^2-1)^2}=\frac{-2x^2+6x-2}{(x^2-1)^2}=\frac{-2(x^2-3x+1)}{(x^2-1)^2}$$

$f'(x)=0$ or $f'(x)$ DNE

$x^2-3x+1=0$

$x^2-1=0$

$x=\dfrac{3\pm\sqrt{9-4}}{2}=\dfrac{3\pm\sqrt{5}}{2}$

$x=\pm1$, but they are not in the domain

$\boxed{\text{Critical numbers are } \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}}$

Example 3

Find the critical numbers of $f(x)=(2x+3)^2\cdot\sqrt[3]{3x+5}$

$$f'(x)=\left((2x+3)^2\right)'\cdot\sqrt[3]{3x+5}+\left(\sqrt[3]{3x+5}\right)'(2x+3)^2$$

$$f'(x)=\left(2(2x+3)\cdot 2\right)\cdot\sqrt[3]{3x+5}+\left(\frac{1}{3}(3x+5)^{-\frac{2}{3}}\cdot 3\right)'(2x+3)^2$$

$$f'(x)=4(2x+3)\cdot\sqrt[3]{3x+5}+\frac{(2x+3)^2}{(3x+5)^{\frac{2}{3}}}$$

Example 3 Continued

$$f'(x)=\frac{4(2x+3)\cdot(3x+5)+(2x+3)^2}{(3x+5)^{\frac{2}{3}}}$$

$$f'(x)=\frac{(2x+3)\cdot\left(4(3x+5)+(2x+3)\right)}{(3x+5)^{\frac{2}{3}}}$$

$$f'(x)=\frac{(2x+3)\cdot(14x+23)}{(3x+5)^{\frac{2}{3}}}$$

$f'(x)=0$ or DNE

$$\boxed{\begin{array}{l}\text{3 critical numbers:}\\ x=-\frac{3}{2},\ x=-\frac{23}{14},\ x=-\frac{5}{3}\end{array}}$$

Example 4

If $f(x)=\cos 2x-2\sin x$, find the critical numbers of f on the interval $[0,\ 2\pi]$

Differentiate $f(x)$:

$f'(x)=-2\sin 2x-2\cos x=0$ or $\cancel{\text{DNE}}$

$4\sin x\cos x+2\cos x=0$

$2\cos x\,(2\sin x+1)=0$

$\cos x=0$ or $\sin x=-\frac{1}{2}$ (quadr. III, IV)

$x=\frac{\pi}{2},\ \frac{3\pi}{2}$ or $x=\frac{7\pi}{6},\ \frac{11\pi}{6}$

Absolute Extrema

$f(c)$ is the absolute minimum value of f if $f(c)\le f(x)$ for every x in the domain of f.

$f(c)$ is the absolute maximum value of f if $f(c)\ge f(x)$ for every x in the domain of f.

To find Absolute Extrema on [a,b]:

1. Find all the critical numbers of f in (a,b)
2. Plug in critical numbers and endpoints of the interval into f
3. The max and the min are the largest and the smallest values of f calculated in #2

Example 5

Find absolute extrema for $f(x)=3x^2-10x+7$ on $[-1,3]$

1. Find all the critical numbers of f in (a,b)

$f'(x)=6x-10=0$ or $\cancel{\text{DNE}}$

$6x=10\Rightarrow x=\frac{5}{3}$

2. Plug in critical numbers and endpoints of the interval into f

$f(-1)=20$

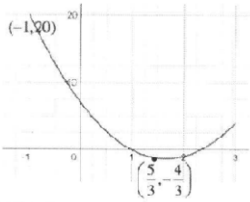
$f(5/3)=-\frac{4}{3}$

$f(3)=4$

3. The max and the min are the largest and the smallest values of f calculated in #2.

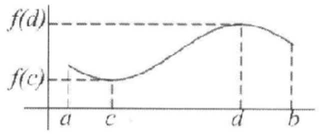
The absolute max is $f(-1)=20$

The absolute min is $f(5/3)=-\frac{4}{3}$



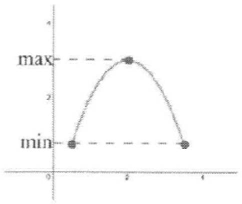
Extreme Value Theorem

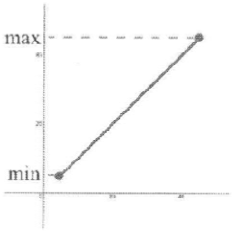
The Extreme Value Theorem states that if *f* is a continuous function on the closed interval [*a*,*b*], then *f* attains a maximum value and a minimum value at least once in [*a*,*b*].



f is continuous on closed interval [*a*,*b*]. Therefore *f* has a minimum and maximum value at *f*(*c*) and *f*(*d*).

A couple Other Cases:





Example 6

What are the maximum and minimum values of $f(x)=x^4-3x^3-1$ on $[-2,2]$?

f is continuous on $[-2,2]$

$f'(x)=4x^3-9x^2$

$4x^3-9x^2=0$

$x=0,\frac{9}{4}$

$\frac{9}{4}$ is not in the interval $[-2,2]$

$x=0$ is the only critical point on the interval $[-2,2]$

Test endpoints and critical points:

$f(-2)=(-2)^4-3(-2)^3-1$
$=39 \Leftarrow$ Maximum

$f(0)=(0)^4-3(0)^3-1$
$=-1$

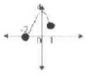
$f(2)=(2)^4-3(2)^3-1$
$=-9 \Leftarrow$ Minimum

Example 7

For $f(x)=4-x^2$, find the absolute extrema of f on the following intervals:

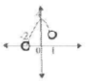
a) Closed interval $[-2,1]$

Max : $f(0)=4$
Min : $f(-2)=0$



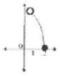
b) Open interval $(-2,1)$

Max : $f(0)=4$
Min : none



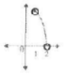
c) $(1,2]$

Max : none
Min : $f(2)=0$



d) $(1,2)$

Max : none
Min : none



Difference between the Extreme Value and the Location of the Extreme Value.

- **If you are asked to "find a minimum or a maximum of a function", they want *y*-value.**

- **If you are asked "at which *x* does this function have a min or max", they want *x*-value.**

- **If you are asked "at which point does this function have a min or max", they want both *x* and *y* coordinates.**

Example 8

Find the points of extrema of $f(x)=5-6x^2-2x^3$ on $[-3,1]$.

$f'(x)=-12x-6x^2$

$f'(x)=0$ or ~~DNE~~

$f'(x)=-12-12x$

$f'(x)=-6x(2+x)$

$x=0,x=-2$

Plug in critical numbers and endpoints into f and compare.

$f(0)=5-6(0)^2-2(0)^3=5$

$f(-2)=5-6(-2)^2-2(-2)^3=-3$

$f(-3)=5-6(-3)^2-2(-3)^3=5$

$f(1)=5-6(1)^2-2(1)^3=-3$

Maximum at $(0,5),(-3,5)$
Minimum at $(-2,-3),(1,-3)$

3.2
Rolle's Theorem and Mean Value Theorem

Rolle's Theorem

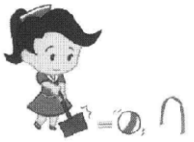
Suppose that $y = f(x)$ is continuous on the closed interval $[a,b]$

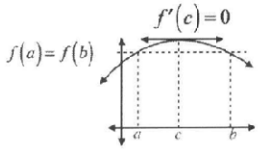
and differentiable on the open interval (a,b).

If $f(a) = f(b)$,

then there is at least one number c between a and b at which

$f'(c) = 0$.





Example 1

Let $f(x) = x^3 - 4x$. Show that f satisfies the hypotheses of Rolle's theorem on the interval $[-2,2]$, and find all real numbers c in the open interval $(-2,2)$, such that $f'(c) = 0$

$f(x) = x^3 - 4x$ is continuous for all x on the interval $[-2,2]$

and differentiable for x on $(-2,2)$

$f(x)$ is a polynomial $\Rightarrow$ continuous and differentiable.

$f(-2) = (-2)^3 - 4\cdot(-2) = 0$

$f(2) = 2^3 - 4\cdot 2 = 0$

Hypotheses of Rolle's Theorem are satisfied and $f(-2) = f(2)$

Thus: $f'(c) = 3c^2 - 4 = 0$ at least once between -2 and 2.

$$\boxed{c_1 = \frac{2\sqrt{3}}{3} \text{ and } c_2 = -\frac{2\sqrt{3}}{3}}$$

Mean Value Theorem

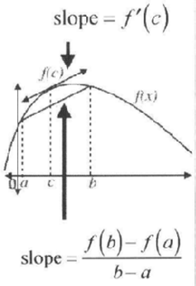
If $y = f(x)$ is continuous on the closed interval $[a,b]$
and differentiable on the open interval (a,b), then
there is at least one number c between a and b at which

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Geometrical Illustration of the Mean Value Theorem

If $y = f(x)$ is continuous on the closed interval $[a,b]$
and differentiable on the open interval (a,b), then
there is at least one number c between a and b at which

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$



$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Example 2

The graph of $y = f(x)$ on the closed interval $[-4,8]$ is shown in the figure below. If f is continuous on $[-4,8]$ and differentiable on $(-4,8)$, then there exists a c, $-4 < c < 8$, such that

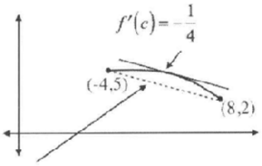
$(A)\ f'(c) = -4$

$(B)\ f(c) = -\frac{1}{4}$

$(C)\ f'(c) = \frac{1}{4}$

$(D)\ f'(c) = -\frac{1}{4}$

$(E)\ f(c) = 0$



By the Mean Value Theorem,

$f'(c) = -\frac{1}{4}$

$$\frac{f(b)-f(a)}{b-a} = \frac{2-5}{8-(-4)} = \boxed{-\frac{1}{4}}\ (D)$$

Example 3

ff $f(x) = \frac{1}{3}x^2 + 2$, show that f satifies the hypotheses of the Mean Value Theorem on the interval $[-1,3]$, and find a number c in $(-1,3)$ that satisfies the conclusion of the theorem.

Solution

$f(x)$ is a Polynomial $\Rightarrow$

f is continuous on $[-1,3]$,

f is differentiable on $(-1,3)$.

$$\frac{f(3)-f(-1)}{3-(-1)} = f'(c)$$

Example 3 Continued

$f'(x)=\frac{2}{3}x,$

$f(3)=5, f(-1)=2\frac{1}{3}$

$\frac{5-2\frac{1}{3}}{4}=\frac{2}{3}c$

$\frac{\frac{8}{3}}{4}=\frac{2}{3}c$

$\frac{2}{3}=\frac{2}{3}c$

$\boxed{c=1}$ where $-1<1<3$

Example 4

f $f(x)=x^3-4x+3$, show that f satisfies the hypotheses of the Mean Value Theorem on the interval $[1,5]$,and find a number c in he open interval$(1,5)$ that satisfies the conclusion of the theorem.

Solution

$f(x)$ is a Polynomial, so

f is continuous on $[1,5]$,

f is differentiable on $(1,5)$.

$\frac{f(5)-f(1)}{5-1}=f'(c)$

$f(5)=108, f(1)=0$

$f'(x)=3x^2-4$

$\frac{108-0}{4}=3c^2-4$

$27=3c^2-4$

$c^2=\frac{31}{3}\Rightarrow c=\pm\sqrt{\frac{31}{3}}$

Since $-\sqrt{\frac{31}{3}}$ is not in the interval $(1,5)$, the answer is $\boxed{\sqrt{\frac{31}{3}}}$

Example 5

Let $f(x)$ be a differentiable function defined on the interval $-5\le x\le 5$. The table below gives the value of $f(x)$ and its dervative $f'(x)$ at several points of the domain.

x	-5	-4	-2	0	2	4	5
f(x)	45	25	13	3	2	4	5
f'(x)	-5	-4	-3	-1	0	1	0

At what point does the line tangent to the graph of $f(x)$ and parallel to the segment between the endpoints intersects the x-axis?

Example 5 Continued **Solution**

$f(x)$ is differentiable function, so it continuous. By MVT there s at least one point at which the line tangent to the graph of $f(x)$ s parallel to the segment between the endpoints.

$\text{slope}=\frac{5-45}{5-(-5)}=\frac{-40}{10}=-4.$

From the table: the corresponding point is $(-4,25)$

Equation of tangent line: $y-y_0=m(x-x_0)$

$y-25=-4[x-(-4)]\Rightarrow y=-4x+9$

To find x-intercept, set $y=0$

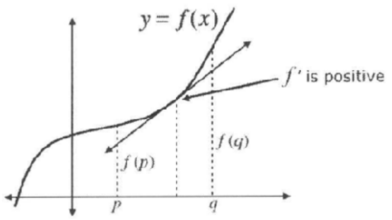
$0=-4x+9\Rightarrow x=\frac{9}{4}$

$\boxed{\left(\frac{9}{4},0\right)}$

3.3
The First Derivative Test.
Using First Derivative in Graphing

Increasing function

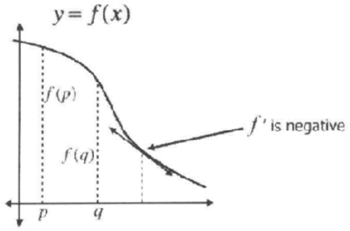
A function f **is increasing** on an interval I if $f(p) < f(q)$ for all p and q in I with $p < q$



When $f(x)$ is increasing $f'(x)$ is positive

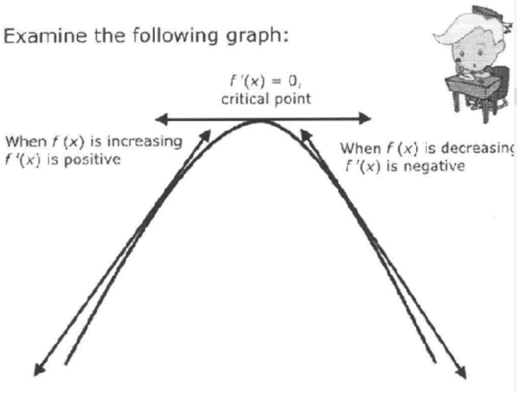
Decreasing function

A function f **is decreasing** on an interval I if $f(p) > f(q)$ for all p and q in I with $p < q$



When $f(x)$ is decreasing $f'(x)$ is negative

Examine the following graph:



Example 1

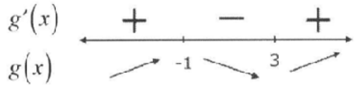
Let $g(x) = x^3 - 3x^2 - 9x + 22$

a) Find the intervals on which g is increasing and decreasing.

$$g'(x) = 3x^2 - 6x - 9$$

$$3(x+1)(x-3) = 0$$

$$x = -1,\ 3 \text{ are critical numbers}$$



g(x) decreases on: [-1 , 3]

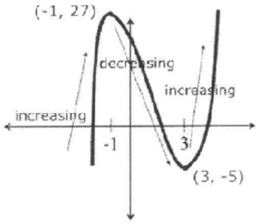
g(x) increases on: (-∞, -1] & [3 , ∞)

Example 1 Continued

Let $g(x) = x^3 - 3x^2 - 9x + 22$

b) Sketch this graph

$g(-1) = 27$ and $g(3) = -5$

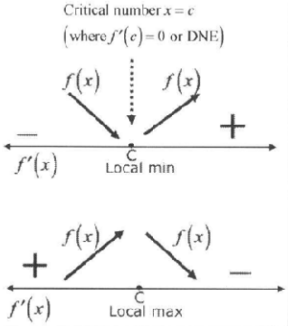


The First-Derivative test

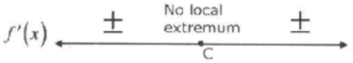
Let f be continuous at c and differentiable on the open interval containing c except possibly at c itself

Local minimum occurs at $x = c$, where $f'(x)$ changes from negative to positive.

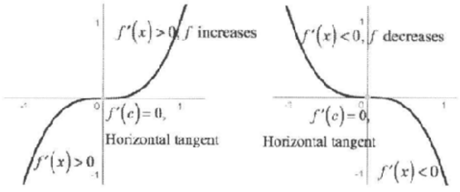
Local maximum occurs at $x = c$, where $f'(x)$ changes from positive to negative .



What happens when $f'(c) = 0$, but $f'(x)$ does not change sign around $x = c$?



For Example:



No local extremum at x=c

Example 2

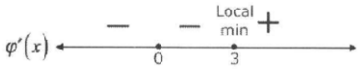
Let $\varphi(x)=x^4-4x^3+12$

a) Find and classify the critical numbers of φ

$\varphi'(x)=4x^3-12x^2$

$\varphi'(x)=4x^2(x-3)=0$

$x=0$ and $x=3$ are critical numbers



Local minimum occurs at $x=3$, where $f'(x)$ changes from negative to positive.

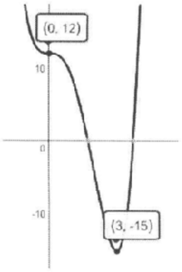
No extremum at $x=0$ and local(or relative) minimum at $x=3$

Example 2 Continued

Let $\varphi(x)=x^4-4x^3+12$

b) Graph φ

$\varphi(0)=12$ and $\varphi(3)=-15$



Notice that $\varphi(3)=-15$ is the lowest value of φ for all x, not only near $x=3$. So, the absolute minimum value of $\varphi(x)$ is -15 at $x=3$. There is no absolute maximum.

Example 3. Find the points of local extrema of f and the intervals on which f is increasing and decreasing, and sketch the graph of f.

$$f(x)=x^{2/3}(x-7)^2+2$$

$$f'(x)=x^{2/3}\left((x-7)^2\right)'+\left(x^{2/3}\right)'(x-7)^2$$

$$=x^{2/3}\left(2(x-7)\right)+\left(\frac{2}{3}x^{-1/3}\right)(x-7)^2$$

$$=2x^{2/3}(x-7)+\frac{2(x-7)^2}{3x^{1/3}}$$

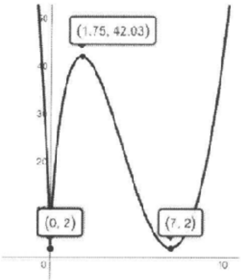
$$=\frac{6x(x-7)+2(x-7)^2}{3x^{1/3}}$$

$$=\frac{(x-7)(6x+2x-14)}{3x^{1/3}}=\frac{2(x-7)(4x-7)}{3x^{1/3}}$$

$f'(x)=0$ or DNE

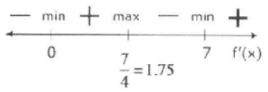
Critical numbers: $x=7,\frac{7}{4},0$

Example 3 Continued



At (0,2) : $f'(x)$ DNE

Test f′(x):

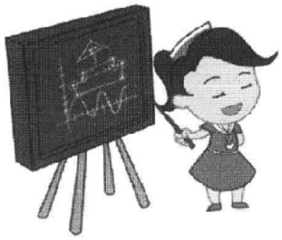


Increasing: $\left[0,\frac{7}{4}\right]\cup[7,\infty)$

Decreasing: $(-\infty,0]\cup\left[\frac{7}{4},7\right]$

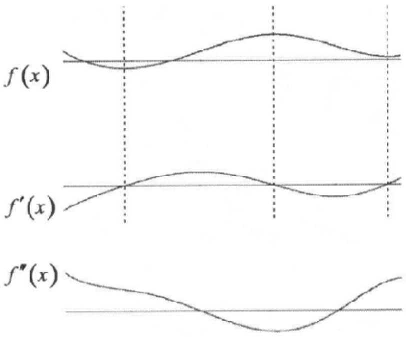
3.4 Concavity and the Second Derivative Test

Let's look at the example of how the graphs $f(x), f'(x)$ and $f''(x)$ are related

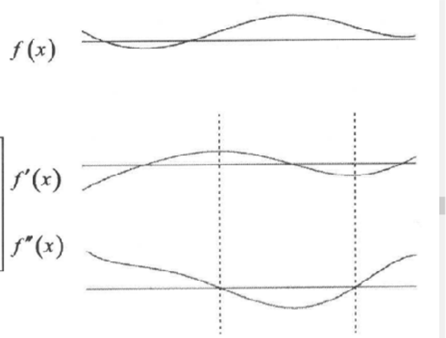


$f(x)$ decreases $\Rightarrow f'(x)<0$ | $f(x)$ increases $\Rightarrow f'(x)>0$ | $f(x)$ decreases $\Rightarrow f'(x)<0$

$f(x)$ and $f'(x)$



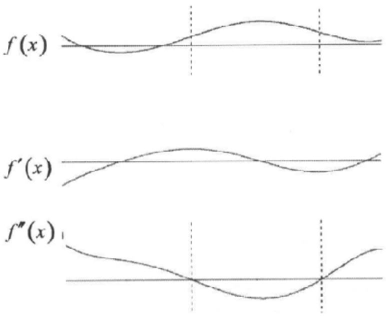
$f'(x)$ and $f''(x)$



$f'(x)$ increases $\Rightarrow f''(x)>0$ | $f'(x)$ decreases $\Rightarrow f''(x)<0$ | $f'(x)$ inc $f''(x)>0$

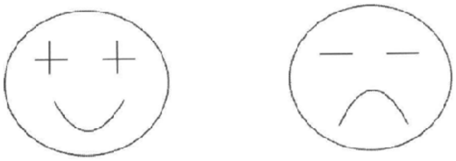
$f''(x)>0$ $f(x)$ concave up | $f''(x)<0$ $f(x)$ concave down | $f''(x)>0$ $f(x)$ concave up

$f(x)$ and $f''(x)$

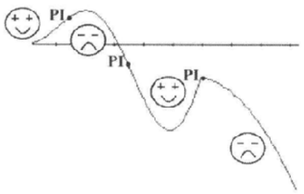


$f''(x)>0$ $f(x)$ concave up

$f''(x)<0$ $f(x)$ concave down



A Point of Inflection occurs at the point where the concavity changes.

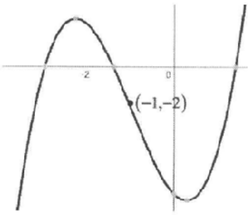


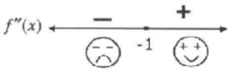
For inflection points, set $f''(x)=0$ or DNE.
If $f''(x)$ changes sign passing through the point, it is a PI.

Example 1

If $f(x)=x^3+3x^2-2x-6$, determine intervals on which the graph of $f(x)$ is concave upward or is concave downward and sketch the graph.

$f'(x)=3x^2+6x-2$
$f''(x)=6x+6=6(x+1)$
$6(x+1)=0$ or ~~DNE~~
$x=-1$
$f(-1)=-1+3+2-6=-2$



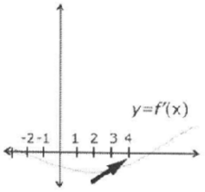


Concave down: $(-\infty,-1)$ Concave up: $(-1,\infty)$

Example 2

The graph of $f'(x)$ is shown below. On which of the following intervals is the graph of $f(x)$ concave up?

a)(-2,4) b) (2,4) c) (-2,2) d) (0,2) e) (-2,0)



Solution

$f(x)$ concave up

→ $f''(x)=[f'(x)]' >0$

It means $f'(x)$ is increasing.

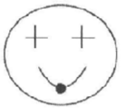
$f'(x)$ is increasing on [2,4]

$\boxed{f(x) \text{ is concave up on } (2,4)}$

2nd Derivative Test for Local Max and Min

Suppose that f is differentiable on an open interval containing c and that $f'(c)=0$

$\boxed{\text{If } f''(c)>0,\ f \text{ has a minimum at } c}$



$\boxed{\text{If } f''(c)<0,\ f \text{ has a maximum at } c}$

If $f'(c)$ DNE, we don't use 2nd derivative test because $f''(c)$ would not exist and there will be no conclusion

Example 3

If $f(x)=x^{2/3}(x+4)$, find the local extrema, discuss concavity, and find points of inflection.

$$\boxed{f(x)=x^{5/3}+4x^{2/3}}$$

$$f'(x)=\frac{5}{3}x^{2/3}+\frac{8}{3}x^{-1/3}=\frac{5x^{2/3}}{3}+\frac{8}{3x^{1/3}}+=\frac{5x+8}{3x^{1/3}}$$

$$\boxed{f'(x)=\frac{1}{3}\left(\frac{5x+8}{x^{1/3}}\right)}$$

$$f''(x)=\frac{10}{9}x^{-1/3}+\frac{-8}{9}x^{-4/3}=\frac{2}{9}\left(5x^{-1/3}-4x^{-4/3}\right)=\frac{2}{9}\left(\frac{5}{x^{1/3}}-\frac{4}{x^{4/3}}\right)$$

$$\boxed{f''(x)=\frac{2}{9}\left(\frac{5x-4}{x^{4/3}}\right)}$$

Example 3 Continued

$f'(x)=\frac{1}{3}\left(\frac{5x+8}{x^{1/3}}\right)=0$ or DNE

$\boxed{f''(x)=\frac{2}{9}\left(\frac{5x-4}{x^{4/3}}\right)}$

The critical numbers are x = -8/5 , 0.

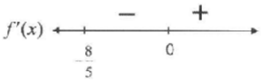
Use second derivative test for x=-8/5

$f''\left(-\frac{8}{5}\right)<0$ ☹ **max** $f\left(-\frac{8}{5}\right)\approx 3.28$

$\boxed{\text{Local max } \left(-\frac{8}{5},3.28\right)}$

If $f'(0)$ DNE, we do not use 2nd derivative test for $x=0$

Use 1st derivative test.



Local min at $x=0$
$f(0)=0$,
f is not differentiable at $(0,0)$

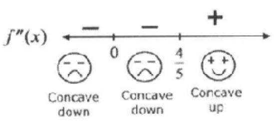
Example 3 Continued

$\boxed{f''(x)=\frac{2}{9}\left(\frac{5x-4}{x^{4/3}}\right)}$

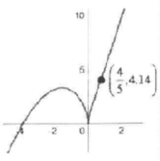
$\boxed{\text{Local max } \left(-\frac{8}{5},3.28\right)}$

Local min at $x=0$
$f(0)=0$,
f is not differentiable at $(0,0)$

Inflection points could be the points where $f''(x)=0$ or DNE. So, check $x=0,\frac{4}{5}$



$\left(\frac{4}{5},4.14\right)$ is PI



Concave down on $(-\infty,0)\cup\left(0,\frac{4}{5}\right)$
Concave up on $\left(\frac{4}{5},\infty\right)$

How to Sketch a Graph Based on Information of $f'(x)$ and $f''(x)$

Example 4

Given: $f(0)=4;\ f(2)=2;\ f(5)=6.5$

$f'(0)=f'(2)=0$

$f'(x)>0$ if $|x-1|>1$

$f'(x)<0$ if $|x-1|<1$

$f''(x)<0$ if $x<1$ or if $|x-4|<1$

$f''(x)>0$ if $|x-2|<1$ or if $x>5$

Example 4 Continued

Solution

$f'(x) > 0$ if $|x-1| > 1$

f increases: $x-1>1$ or $x-1<-1$

$x>2$ or $x<0$

$f'(x) < 0$ if $|x-1| < 1$

f decreases: $-1 < x-1 < 1$

$0<x<2$

☹ $f''(x) < 0$ if $x<1$ or if $|x-4|<1$

$-1 < x-4 < 1$

$3<x<5$

☺ $f''(x) > 0$ if $|x-2|<1$ or if $x>5$

$-1<x-2<1$

$1<x<3$

Example 4 Continued

Plot the pts: $f(0)=4; f(2)=2; f(5)=6.5$

$f'(0)=f'(2)=0 \Rightarrow$ Horizontal tangents at $x=0,2$.

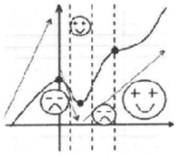
f increases: $x>2$ or $x<0$

f decreases: $0<x<2$

f concaves down if $x<1$ or $3<x<5$

f concaves up if $1<x<3$ or $x>5$

Graph:



3.5 Summary of Graphical Methods. Guidelines for sketching the Graph of a Function using First and Second Derivatives.

Guidelines for sketching a graph of y = f(x)

There are seven guidelines to follow while sketching a graph.

Example 1. Discuss and sketch the graph of $f(x)=\dfrac{3x^2}{16-x^2}$

1. Domain – find all x that are possible.

The denominator of a function cannot equal 0.

$16-x^2 \neq 0$

$x^2 \neq 16$

$x \neq -4, 4$

The domain of f consists of all real numbers except –4 and 4.

$\boxed{D=\{x \mid x \neq -4,4\}}$

2. Continuity – determine whether f is continuous on its domain.

f is a rational function, it is continuous on its domain,

but f has infinite discontinuities at –4 and 4 which are not on the domain

. Intercepts: find the x-and y-intercepts.

-intercepts: $x=0$ or find $f(0)$

$y=f(0)=\dfrac{3(0)^2}{16-(0)^2}=0$ $\boxed{(0,0)}$

-intercepts: $y=0$ or

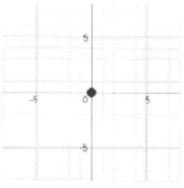
solve $f(x)=0$

$f(x)=\dfrac{3x^2}{16-x^2}=0$

$x^2=0 \Rightarrow x=0$ $\boxed{(0,0)}$

The graph intersects both axes at the origin.

$f(x)=\dfrac{3x^2}{16-x^2}$



4. Symmetry – check to see if the function is odd or even.

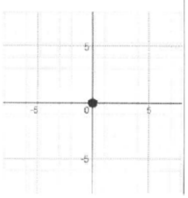
If the function is even, the graph will be symmetrical with respect to the y-axis.

If the function is odd, the graph will be symmetrical with respect to the origin.

$f(x)=\dfrac{3x^2}{16-x^2}$

$f(-x)=\dfrac{3(-x)^2}{16-(-x)^2}=\dfrac{3x^2}{16-x^2}=f(x)$

Since $f(-x)=f(x)$, this function is even and is symmetrical with respect to the y-axis.



5 a.Critical Numbers – find values of x on the domain of $f(x)$ where $f'(x)=0$ or $f'(x)$ DNE

$$f'(x)=\frac{(3x^2)'(16-x^2)-(16-x^2)'(3x^2)}{(16-x^2)^2}$$

$$=\frac{(6x)(16-x^2)-(-2x)(3x^2)}{(16-x^2)^2}$$

$$=\frac{(96x-6x^3)-(-6x^3)}{(16-x^2)^2}=\frac{96x}{(16-x^2)^2}$$

$f'(x)=0$ or $f'(x)$ DNE

$\frac{96x}{(16-x^2)^2}=0$ or $(16-x^2)^2=0$

$\boxed{x=0\text{, critical number}}$

$x=-4$ and $4 \Rightarrow$ Not on the domain.
Not critical numbers.
But increasing/decreasing
could change around them.

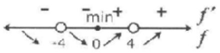
5 b. Local Extrema

Use First Derivative test to find local extrema.

Increasing/decreasing

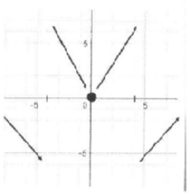
Increasing if $f'(x)>0$, decreasing if $f'(x)<0$

$f(x)=\frac{3x^2}{16-x^2}$



$\boxed{\begin{array}{l}\text{Increasing: } [0,4)\cup(4,\infty)\\ \text{Decreasing: } (-\infty,-4)\cup(-4,0]\end{array}}$

$\boxed{\text{Local min at } x=0:(0,0)}$



6 a. Points of Inflection

Find $f''(x)$, set $f''(x)=0$ and $f''(x)=DNE$,

Find x where concavity of $f(x)$ changes

$$f''(x)=\left[\frac{96x}{(16-x^2)^2}\right]'$$

$$=96\cdot\frac{x'\cdot(16-x^2)^2-((16-x^2)^2)'\cdot x}{(16-x^2)^4}$$

$$=96\cdot\frac{(16-x^2)^2-2(16-x^2)\cdot(-2x)\cdot x}{(16-x^2)^4}$$

$$=96\cdot\frac{\cancel{(16-x^2)}((16-x^2)+4x^2)}{(16-x^2)^{\cancel{4}3}}=\frac{96(16+3x^2)}{(16-x^2)^3}$$

$f''(x)\neq 0$ or $f''(x)$ DNE

$(16-x^2)^3=0$

$x=-4,4$

These are not PIs,
not on the domain,
but concavity could
change around them.

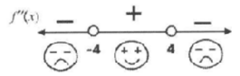
$\boxed{\text{No points of inflection.}}$

6 b. Concavity

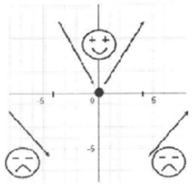
If $f''(x)>0$, graph is concave up

If $f''(x)<0$, graph is concave down.

$f(x)=\frac{3x^2}{16-x^2}$



$\boxed{\begin{array}{l}\text{Concave down}\\ \text{on } (-\infty,-4)\cup(4,\infty)\end{array}}$ $\boxed{\begin{array}{l}\text{Concave up}\\ \text{on } (-4,4)\end{array}}$



7. Asymptotes: lines graph is approaching to.

7a. Horizontal asymptote: if $\lim_{x\to\infty} f(x)=L$ or if $\lim_{x\to-\infty} f(x)=L$, then the line $y=L$ is a horizontal asymptote.

$f(x)=\frac{3x^2}{16-x^2}$

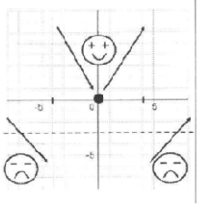
$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{3x^2}{16-x^2}=\lim_{x\to+\infty}\frac{3}{\frac{16}{x^2}-1}=-3$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{3x^2}{16-x^2}=\lim_{x\to-\infty}\frac{3}{\frac{16}{x^2}-1}=-3$$

$\boxed{\text{Horizontal asymptote : } y=-3}$

Or: Power of numerator = power of denominator

so Horizontal asymptote: $y=\frac{3}{-1}=-3$



7b. Vertical asymptote: If $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ is equal to either $+\infty$ or $-\infty$, then the line $x=a$ is a vertical asymptote.

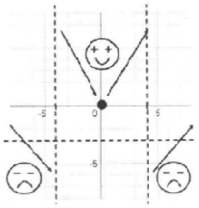
Or, the vertical asymptotes correspond to the zero of the denominator $16-x^2$

$f(x)=\frac{3x^2}{16-x^2}$

$16-x^2=0$

$x=4,\ x=-4$

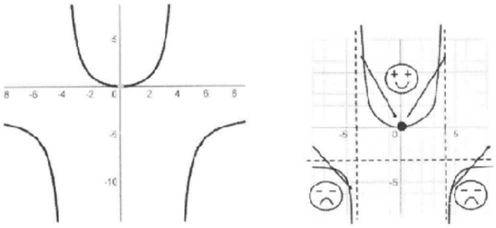
$\boxed{\begin{array}{l}\text{Vertical asymptotes are}\\ x=-4 \text{ and } x=4\end{array}}$



Time to graph!

$$f(x)=\frac{3x^2}{16-x^2}$$

Check using graphing calculator:



Example 2. Discuss and sketch the graph of $f(x)=\frac{x^2-16}{2x-6}$

1.Domain: $2x-6\neq 0$
$x\neq 3$

Domain = all real numbers, except $x=3$

$\boxed{D=\{x|x\neq 3\}}$

2.Continuity

Continuous on its domain, But it has infinite discontinuity at $x=3$

3.Intercepts:

x-intercepts: $y=0$ $\quad x^2-16=0$

$\boxed{x=-4, 4}$

y-intercepts: $x=0$

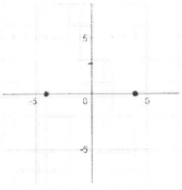
$$y=\frac{0^2-16}{2(0)-6} \qquad \boxed{y=\frac{8}{3}}$$

4.Symmetry:

$f(-x)\neq f(x)$ or $-f(x)$

$f(x)$ is neither odd nor even.

So $f(x)$ is not symmetric with respect to the y-axis or origin.



5.Critical Numbers and Extrema:

$$f(x)=\frac{x^2-16}{2x-6}$$

$$f'(x)=\frac{(x^2-16)'(2x-6)-(2x-6)'(x^2-16)}{(2x-6)^2}$$

$$=\frac{2x(2x-6)-2(x^2-16)}{(2x-6)^2}=\frac{2x^2-12x+32}{(2x-6)^2}$$

5.Critical Numbers and Extrema (continued):

$f'(x)\neq 0$, it is >0 since Discriminant of numerator $D=144-4\cdot 2\cdot 32<0$

$\boxed{f(x) \text{ increases}}$ for all x on the domain

$f'(x)$ DNE $\Rightarrow x=3$ but not on domain, not a critical number. No change in increasing/decreasing around $x=3$ because $f(x)$ increases.

There are no critical numbers, no local extrema.

.Points of Inflection and Concavity:

$$f''(x)=\left(\frac{2x^2-12x+32}{(2x-6)^2}\right)'=\left(\frac{2(x^2-6x+16)}{4(x-3)^2}\right)'$$

$$=\frac{1}{2}\cdot\frac{(x^2-6x+16)'(x-3)^2-((x-3)^2)'(x^2-6x+16)}{(x-3)^4}$$

$$=\frac{1}{2}\cdot\frac{(2x-6)(x-3)^2-2(x-3)(x^2-6x+16)}{(x-3)^4}$$

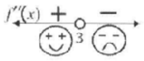
$$=\frac{(x-3)^3-(x-3)(x^2-6x+16)}{(x-3)^4}=-\frac{7}{(x-3)^3}$$

6.Points of Inflection and Concavity(continued):

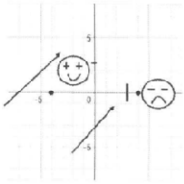
$f''(x)\neq 0$ or $f''(x)$DNE

$$(x-3)^3=0\Rightarrow x=3$$

$x=3$ is not a PI, not on the domain, but concavity could change around it.



Concave up on $(-\infty,3)$
Concave downon $(3,\infty)$



7.Asymptotes:

$$f(x)=\frac{x^2-16}{2x-6}$$

Power of numerator is greater than power of denominator, so use long division to find the slant asymptote.

$$\begin{array}{r|l} & .5x+1.5 \quad R:-7 \\ 2x-6 & x^2-16 \\ & \underline{x^2-3x} \\ & \quad 3x-16 \\ & \quad \underline{3x-9} \\ & \qquad -7 \end{array}$$

No horizontal asymptote.

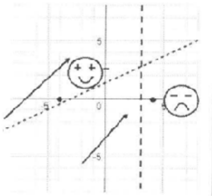
The slant asymptote is $y{-}0.5x+1.5$

$$\frac{x^2-16}{2x-6}=0.5x+1.5-\frac{7}{2x-6}$$

Vertical asymptote: Occurs where denominator of $f(x)$ is 0.

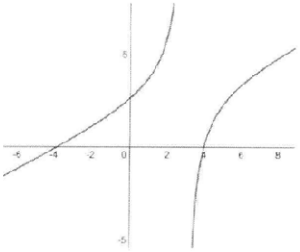
$2x-6=0$

Vertical asymptote is $x=3$

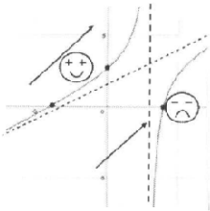


Time to graph!

Check using graphing calculator:



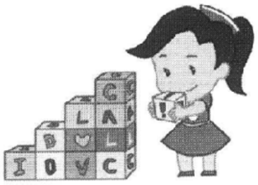
$$f(x)=\frac{x^2-16}{2x-6}$$



3.6 Optimization Problems

Optimization. Applications of Maxima and Minima

The First Derivative may be used to find the largest or smallest value of a function.



Example 1: Find two positive numbers whose sum is 20 and whose product is as large as possible.

Let the 1st number $= x$

Then 2nd number $= (20 - x)$

product $F(x)= x(20-x)=20x-x^2$

$F'(x)=20-2x=2(10-x)$

$F'(x)=0 \Rightarrow x=10$

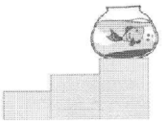
1st number = 10,
2nd number = 20-10 = 10

Compare the product *F*(x) at the endpoints and at x = 10:

$F(0)=0$

Maximum $\boxed{F(10)=100}$

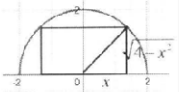
$F(20)=0$



Example 2: A rectangle is inscribed in a semicircle of radius 2cm. What is the largest area the rectangle can have and its dimensions?

Let length $= 2x \Rightarrow$ Height $=\sqrt{4-x^2}$

Area $A(x)=2x\sqrt{4-x^2}$



$$A'(x)=\left(2x\sqrt{4-x^2}\right)'=2\left(x'\sqrt{4-x^2}+\left(\sqrt{4-x^2}\right)'x\right)$$

$$=2\left(\sqrt{4-x^2}+\frac{1\cdot(-2x)}{2\sqrt{4-x^2}}\cdot x\right)$$

$$=2\left(\frac{(4-x^2)-x^2}{\sqrt{4-x^2}}\right)=2\left(\frac{4-2x^2}{\sqrt{4-x^2}}\right)=0$$

Example 2 Continued

$4-2x^2=0$

$x=\pm\sqrt{2} \Rightarrow$ Since $0 \le x \le 2$, then $\boxed{x=\sqrt{2}}$

Compare the values for the area at the endpoints and at $x=\sqrt{2}$

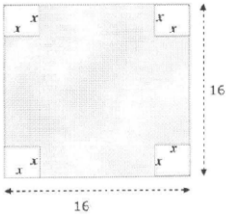
$A(0)=0,$

$A(2)=0$

$A\left(\sqrt{2}\right)=2\sqrt{2}\cdot\sqrt{2}=4$

$A\left(\sqrt{2}\right)=4\,\text{cm}^2$ is the largest area.
Rectangle is $2\sqrt{2}$ by $\sqrt{2}$

Example 3: A square sheet of tin *16* inches on a side is to make a open-top box by cutting a small square of tin from each corner and bending up the sides. How large a square should be cut from each corner to make the box have as large a volume as possible.



$V(x)=x(16-2x)^2$

$V(x)=x\left(16^2-64x+4x^2\right)$

$V(x)=256x-64x^2+4x^3$

$V'(x)=256-128x+12x^2$

$=(16-6x)(16-2x)$

$x=8,\ \frac{8}{3}$

$0<x<8$ so $x=\frac{8}{3}$

$V(0)=0, V(8)=0, V\left(\frac{8}{3}\right)>0$

Dimensions are 8/3 by 8/3

Example 4: A cylindrical metal jar, open at the top, is to have a capacity of 36π in^3. The cost of the material used for the bottom of the jar is 10 cents per in^2, and that of the material used for the curved part is 5 cents per in^2. What are dimensions that will minimize the cost of the material.

Solution

Cost of container = 10 (area of base) + 5 (lateral area)

$C=10\left(\pi r^2\right)+5(2\pi rh)=10\pi\left(r^2+rh\right)$

Express C in one variable: $\pi r^2 h=36\pi \Rightarrow h=\frac{36}{r^2}$

Plug into C: $C=10\pi\left(r^2+r\cdot\frac{36}{r^2}\right)=10\pi\left(r^2+\frac{36}{r}\right)$

Find derivative: $C'=10\pi\left(2r-\frac{36}{r^2}\right)=20\pi\left(r-\frac{18}{r^2}\right)$

Set derivative = 0

Example 4 Continued

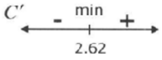
$C'=0$

$20\pi\left(r-\frac{18}{r^2}\right)=0$

$r^3=18$

$r=\sqrt[3]{18}\approx 2.62$

Check if r = 2.62 is at the absolute minimum



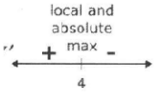
r = 2.62 is at the absolute minimum

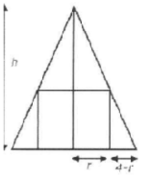
because the function is decreasing and then increasing

$h=\frac{36}{r^2}=\frac{36}{2.62^2}\approx 5.24$

Radius by height: 2.62 in by 5.24 in

xample 5: Find the maximum volume of a right circular linder that can be inscribed in a right circular cone of titude 24 centimeters, and base radius 6 centimeters, if the xes of the cylinder and cone coincide.





Volume of cylinder: $V=\pi r^2 h$

Express V in one variable r.

Use similar triangles:

$\frac{h}{6-r}=\frac{24}{6}=4 \Rightarrow h=4(6-r)$

Plug in:

$V=\pi r^2\cdot 4(6-r)=4\pi\left(6r^2-r^3\right)$

Example 5 Continued

Differentiate:

$V'=4\pi\left(12r-3r^2\right)$

$=4\pi\cdot 3r(4-r)=12\pi r(4-r)$

Set $V'=0$

$12\pi r(4-r)=0$

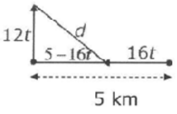
$r=4$ and $r\neq 0$

$V(4)=\pi\cdot 4^2\cdot 4(6-4)=\boxed{128\pi}$

Example 6

t 9 AM, Fin started biking north from Mathland at 12km/h. t the same time, Inty is is 5km east of Mathland biking west t 16km/h. Express the distance d between Fin and Inty as a unction of the time t in hours after 9 AM. At what time are in and Inty closest to each other and what is the minimum istance?

$d(t)=?\quad t_{\min}=?\quad d_{\min}=?$



Solution

$$d(t)=\sqrt{(12t)^2+(5-16t)^2}$$

$$=\sqrt{144t^2+25-160t+256t^2}$$

$$\boxed{d(t)=\sqrt{400t^2-160t+25}}$$

Example 6 Continued

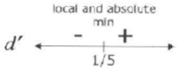
Find the minimum of the function using d′(t) :

$$d'(t)=\frac{1}{2\sqrt{400t^2-160t+25}}\cdot(800t-160)$$

$d'(t)=0$ or $d'(t)$ DNE

$800t-160=0$ Discriminant $D=b^2-4ac<0$

$t=\frac{1}{5}$ hours = 12 min $\Rightarrow$ Denominator $\neq 0$



$$d\left(\frac{1}{5}\right)=\sqrt{400\cdot\left(\frac{1}{5}\right)^2-160\cdot\left(\frac{1}{5}\right)+25}=\sqrt{16-32+25}=3$$

Fin and Inty are closest at 9:12 AM

Minimum distance is 3 km

3.7
Rectilinear Motion

Rectilinear Motion

If a point P is moving along a line l, its motion is **rectilinear.**

Definitions

Let $s(t)$ be the coordinate of a point P on a coordinate line l at time t.

The velocity of P is $v(t)=s'(t)$

The speed of P is $|v(t)|$.

The acceleration of P is $a(t)=v'(t)=s''(t)$.

$v(t)>0\Rightarrow s'(t)>0\Rightarrow s(t)$ is increasing

$\Rightarrow$ point P is moving in the positive direction on l

$v(t)<0\Rightarrow s'(t)<0\Rightarrow s(t)$ is decreasing

$\Rightarrow$ point P is moving in the negative direction on l

$v(t)=0$ where point P changes direction

$a(t)=v'(t)>0\Rightarrow v(t)$ is increasing

$a(t)=v'(t)<0\Rightarrow v(t)$ is decreasing

Example 1

The position function s of a point P on a coordinate line is given b $s(t)=t^3-12t^2+36t+10$ with t in seconds and $s(t)$ in centimeters When is point P moving to the right during time interval $[1,8]$?

Point P is moving to the right $\Rightarrow$ velocity is positive

$v(t)=s'(t)=\left(t^3-12t^2+36t+10\right)'=3t^2-24t+36=0$

$3(t-2)(t-6)=0$

$t=2$ and $t=6$

$v(t)=s'(t)$

Point P is moving to the right for $t\in[1,2)\cup(6,8]$

Example 2

A projectile is fired straight upward. Its distance above the ground after t seconds is $s(t) = -16t^2 + 320t$.

a) Find time, velocity and speed at which the projectile hits the ground.

Projectile hits the ground:

$s(t) = -16t^2 + 320t = 0$

$-16t(t-20) = 0 \Rightarrow t = 0$ and $\boxed{t = 20}$

$v(t) = s'(t) = -32t + 320$

$v(20) = -32(20) + 320 = \boxed{-320 \text{ ft/sec}}$

$v(20) < 0 \Rightarrow$ projectile was moving downwards

The speed at $t = 20$ is $|v(20)| = |-320| = \boxed{320 \text{ ft/sec}}$

Example 2 Continued

$s(t) = -16t^2 + 320t.$

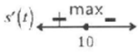
b) Find the maximum altitude achieved by the projectile.

c) Find the acceleration at any time t.

b) To find Max altitude set $s'(t) = 0$

$s'(t) = v(t) = -32t + 320 = 0$

$t = 10$



Max altitude is

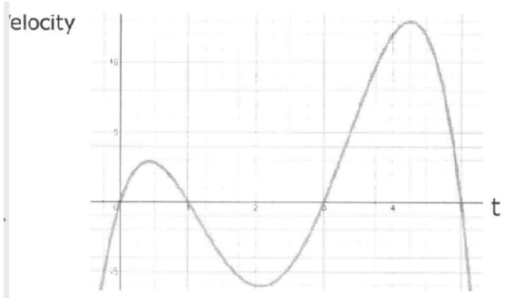
$s(10) = -16(10)^2 + 320(10) = \boxed{1600\,ft}$

Local max = absolute max since $s(t)$ increases, then decreases

c) The acceleration at any time t is

$a(t) = v'(t) = \boxed{-32 \text{ ft/sec}^2}.$

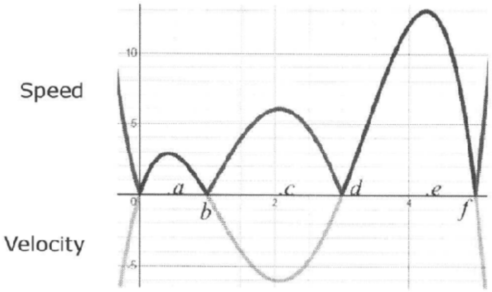
This constant acceleration is caused by the force of gravity.



Speed is the absolute value of velocity.

When the velocity graph is moving away from the t-axis, or absolute value of the velocity increases, the speed increases too

Now let's graph the Speed. The graph of Speed is the one that is above the x-axis.



Increasing/Decreasing Speed

et's fill out this table based on the signs of the Velocity and cceleration (which is the slope of the Velocity). Then conclude if peed is increasing or decreasing on each interval.

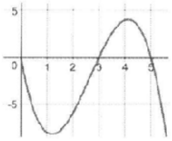
Interval	Velocity + or -	Acceleration + or -	Speed incr/ decreasing
[0,a]	positive	positive	Increasing
[a,b]	positive	negative	Decreasing
[b,c]	negative	negative	Increasing
[c,d]	negative	positive	Decreasing
[d,e]	positive	positive	Increasing
[e,f]	positive	negative	Decreasing

How to determine if the speed is increasing or decreasing:

- If the **velocity and acceleration have the same sign**, then the **speed is increasing**.
- If **velocity and acceleration have different signs**, the **speed is decreasing**.
- If the velocity graph is moving away from the t-axis the speed is increasing .
- If the velocity graph is moving toward the t-axis the speed is decreasing .

Example 3

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. On the interval $2 < t < 3$, is speed of the particle increasing or decreasing?



Example 3 Continued

The speed is decreasing on the interval $2 < t < 3$ since on this interval:

1) Velocity $v(t) < 0$ and

2) $v(t)$ is increasing $\Rightarrow v'(t) = a(t) > 0$

So, Velocity and Acceleration have different signs on this interval.

OR: Velocity graph is moving toward the t-axis
$\Rightarrow$ speed decreases

Calculus and Problems that occur in economics.

Cost function: $C(x)$ = cost of producing x units

Average Cost function: $c(x) = \dfrac{C(x)}{x}$
= average cost of producing one unit

Revenue function: $R(x)$ = revenue received for selling x units

Profit function: $P(x) = R(x) - C(x)$ = profit in selling x units

We regard x as a real number, even though this variable may take on only integer values. We always assume $x \ge 0$, since the production of a negative number of units has no practical significance.

Example 4

A manufacturer of sport equipment parts has a monthly fixed cost of \$10,000, a production cost of \$10 per part, and a selling price of \$18 per part.

a) Find $C(x), c(x), R(x)$, and $P(x)$.

The production costs of manufacturing x parts is $10x$.
The total monthly cost $C(x)$ of manufacturing x parts is

$$C(x) = \boxed{10x + 10{,}000}$$

$$c(x) = \frac{C(x)}{x} = \boxed{10 + \frac{10{,}000}{x}}$$

$$R(x) = \boxed{18x}$$

$$P(x) = R(x) - C(x) = 18x - (10x + 10{,}000) = \boxed{8x - 10{,}000}$$

Example 4 Continued

b) How many parts must be manufactured in order to break even?

The break-even point corresponds to a zero profit:

$$P(x) = 0$$

$$8x - 10000 = 0$$

$$8x = 10000, \text{ or } x = 1250$$

To break even it is necessary to produce and sell 1250 parts per month

Example 5

The weekly cost (in dollars) of manufacturing x wooden tables is given by $C(x) = x^3 - 1.5x^2 - 960x - 10$. Each table produced is sold for \$300. What weekly production rate will maximize the profit?

$$\text{Profit} = P(x) = R(x) - C(x)$$

$$= 300x - (x^3 - 1.5x^2 - 960x + 10)$$

$$= -x^3 + 1.5x^2 + 1260x - 10$$

To maximize it, find $P'(x)$ and set it $= 0$

Example 5 Continued

$P'(x) = -3x^2 + 3x + 1260 = 0$

$-3(x^2 - x - 420) = 0$

$-3(x - 21)(x + 20) = 0$

$x = 21,\ \cancel{-20}$

$P''(x) = -6x + 3;\ P''(21) = -6 \cdot 21 + 3 < 0$

By the Second Derivative test,

a maximum profit occurs if 21 tables per week are manufactured and sold.