Middle school math contest Week 3

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Perhaps the simplest but also most important counting principle is tree counting. To put it simply, tree counting counts the number of integers that are in an arithmetic progression. For example, say we have the integers $1,2,3,4,5,6,7$. Clearly, this list has 7 integers. In general, if we have a list$1,2,3,4,5,6,7$, then there are n integers in the list. Now, say we have the integers $4,5,6,7,8,...,9,10$. How many integers are in this list? You might be tempted to say $10-4=6$, but there are actually 7 integers. One way to see this is by subtracting 3 from each number in the sequence. This gives the new list: $1,2,3,4,5,6,7$. Note that we haven't changed the number of integers in the list, only the value of each integer (which doesn't really matter since we only care about the size of the sequence). As we have said earlier, the list $1,2,3,4,5,6,7$ has 7 integers. Many times, our approach is simply to convert a given list into a list of the form $1,2,3,4,5,6,7$. We can then count the number of integers in the list easily. This process is also called tree counting. Examples

ex1 How many integers are there less than 600 but greater than 500?

The smallest integer greater than 500 is 501 and the largest integer less than 600 is 599. THus we are looking for the number of integers in the list $501,502,...,598,599$. If we subtract 500 from each element of this list, we get the new list $1,2,3,...,99$. This list has 99 elements.

ex2 How many integers are in the list $a,a+d,a+2d,...,b$? (A general arithmetic progression)

We can't simply subtract $a-1$ from each term because we won't get a list of consecutive integers. Instead let us subtract$a$ from each term first. This gives us the list$0,d,2d,...,b-a$.

Now divide each term by d. Doing so, of course, does not change the number of terms in the list; only the values are changed. This gives us the list $0,1,2,...,\frac{b-a}{d}$. Now add 1 to every term. This gives us $1,2,3,...,\frac{b-a}{d}+1$. Clearly this list has $\frac{b-a}{d}+1$ terms.

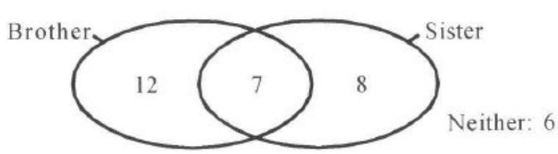
ex3 What is the sum $S=1+2+3+...+n$?

The average of this series is $\frac{1+n}{2}$and there are n terms. So the sum $S=\frac{1+n}{2}\cdot n=\frac{n(n+1)}{2}$.

A Venn diagram can be used to organize counting problems where some items are included in multiple groups and others are excluded. Examples

ex4 In science classroom: 19 students have a brother, 15 have a sister, 7 have both a brother and a sister, and 6 have no siblings at all. How many students are in the classroom?

Working from the inside out, we fill n a 7 where the 2 regions overlap. Then we fill in the rest. We notice 6 students have no siblings for a total of 7+12+8+6 =33 students.

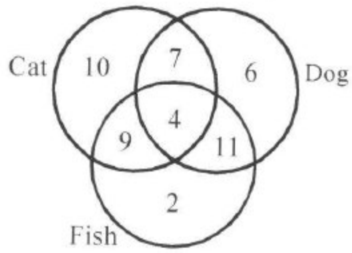


ex5 On a survey of 40 students, 14 students responded that they like Stanford, 18 responded that they like Harvard, and 11 do not like Stanford or Harvard. How many students like both Stanford and Harvard?

Draw your Venn diagram and label the overlap region to have x students. Then$(18-x)+x+(14-x)+11=40$. Solving for $x, x=3$ so there are 3 students who do not like Stanford or Harvard.

ex6 Every student who applied for admission to a veterinary school has at least one pet: 30 have a cat, 28 have a dog and 26 have fish. If 13 students have fish and cat, 15 have fish and dog, 11 have cat and dog, and 4 have cat, dog and fish. How many students applied to the veterinary school?

There are 49 students:



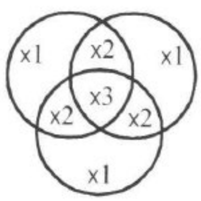
This concept is related to a more general counting tool called Principle of Inclusion - Exclusion (PIE). The general idea is that we first add some stuff, subtract off the stuff that is overcounted, and add back the stuff that was subtracted too many times, then subtract the stuff that was added back too many times, and so on. Let's look at some concrete examples.

ex7 How many positive integers less than or equal to 1000 are divisible by 2 or 3?

The integers divisible by 2 are $2, 4, 6, ..., 1000$. There are 500 of them. The integers divisible by 3 are $3, 6, 9, ..., 999$. There are 333 of them. The answer is not $500 + 333 = 833$ as we have added "too much stuff" since the multiples of 6 are counted both in the list of multiples of 2 and in the multiples of 3. To correct the overcounting, we notice there are 166 numbers in the list $6, 12, 18, ..., 996$ So the answer is $833 - 166 = 667$.

ex8 Repeat ex6 with PIE (Principle of Inclusion-Exclusion)

In a three-set Venn diagram, notice how many times each group gets counted when we add the total members of all 3 sets:



The total number of students is $(30 + 28 + 26) - (13 + 15 + 11) + (4) = 49$.

Handshakes Say we have$n$ people at a party where everyone shakes hands with each other. How many handshakes take place? To simplify things, say the n people have names $1, 2, 3, ..., n$. Then 1 shakes hand with $2, 3, 4, ..., n$. 2 shakes hand with $3, 4, 5, ..., n$ and so on until we have $n - 1$ shakes hand with n. We thus have $(n - 1) + (n - 2) + (n - 3) + ... + 1 = \frac{n(n-1)}{2}$ handshakes. Another way to think is that each of the n people shakes hand with$(n - 1)$ people so the total handshake is $\frac{n(n-1)}{2}$.We divide the product $n(n - 1)$ by 2 because A shakes hand with B and B shakes hand with A is double-counted.

ex9 In a pentagon, how many diagonals are there?

Recall that a pentagon is a polygon with 5 vertices. A diagonal or side is formed ehenever a line segment is drawn between a pair of vertices. It is just handshakes between vertices. Each vertex can shake hand with 2 opposite vertices so the total number of diagonals is $\frac{5(2)}{2} = 5$.

Arithmetic Sequences A sequence is an ordered list of terms. If the sequence is infinite, we usually indicate that the sequence continues forever with 3 dots: $4, 7, 10, 13, ...$ A sequence is called arithmetic if the difference between consecutive terms is always the same. The formula for the n^{th}term $a_n$of an arithmetic sequence with a common difference d is $a_n = a_1 + (n - 1)d$.

ex10 Find the common difference in an arithmetic sequence if the 3^{rd}term is 18 and the 11^{th}term is 44.

The common difference has been added 8 times to get $44 - 18 = 26$. So the common difference is $\frac{26}{8} = 3.25$.

ex11 The 20^{th}term of an arithmetic sequence is 19 and the 50^{th}term is 44. What is the 35^{th}term of the sequence?

$a+19d=19, a+49d=44$ so $2a+68d=63$ or $a+34d=31.5$ which is the 35^{th}term.

Arithmetic Series An arithmetic series is the sum of an arithmetic sequence. To find the sum of the terms, we find the average of the terms and multiple by the number of terms: $a_1+a_2+a_3+...+a_n = n\cdot\frac{a_1+a_n}{2}$

ex12 sum $-9+7+23+39+...+279=?$

This is an arithmetic series with common difference of 16. The average is $\frac{270}{2}=135$. The number of terms is: $\frac{279-(-9)}{16}+1=19$. So the sum is $135\cdot 19=2565$.

Geometric Sequences A geometric sequence is one in which consecutive terms form a common ratio, for example $6, 12, 24, 48, ...$ is a geometric sequence with a common ratio of $12/6=2$; $4, 6, 9, \frac{27}{2}, \frac{81}{4}, ...$ has a common ratio of $\frac{6}{4}$or $\frac{3}{2}$.

For a geometric series with first term a and a common ratio r:

$a, ar, ar^2, ar^3, ar^4, ...$ the n^{th} term is ar^{n-1}.

ex13 The 10^{th}term of the geometric sequence that begins: $54, 36, 24, 26, ...$ can be expressed as $\frac{2^a}{3^b}$. What is the positive difference between a and b.

The common ratio is $\frac{36}{54}=\frac{2}{3}$. The 10^{th}term is $54\cdot(\frac{2}{3})^9=\frac{2^{10}}{3^6}$ So $a-b=4$.

ex14 A geometric sequence has 99 terms, beginning with 12 and ending with 147. What is the 50^{th}term in the sequence?

$a=12, ar^{98}=147$ So $a^2\cdot r^{98}=12\cdot 147=2^2 3^2 7^2, ar^{49}=2\cdot 3\cdot 7=42$ which is the 50^{th}term.

Geometric Series A sum of terms in geometic sequence is called a geometric series.

ex15 Find: $S=\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+...$

Multiply S by $\frac{1}{3}$,the common ratio and we get $\frac{S}{3}=\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+...$ Subtracting this from the original to get $\frac{2S}{3}=\frac{1}{3}$ so $S=\frac{1}{2}$.

ex16 Find $S=a+ar+ar^2+ar^3+...$

$S=\frac{a}{1-r}$.

ex17 Find $S=a+ar+ar^2+ar^3+ar^4+...+ar^{n-1}$

Multiply S by r: $rS=ar+ar^2+ar^3+...+ar^n$. Subtracting from the original equation, we get $(1-r)S=a-ar^n$ So $S=\frac{a(1-r^n)}{1-r}$.

Counting Practices and Homework

1. How many even integers are less than 600 but greater than 500?
2. John starts counting at 130 and counts by fives. What is the $13th$ number that John says?
3. What is the value of $4+7+10+...+301$?
4. What is the value of $a+(a+d)+(a+2d)+(a+3d)+...+(a+(n-2)d)+(a+(n-1)d)$?
5. You are assigned even problems from 10 through 40 for homework. How many problems are there?
6. Paul is making a ruler. He places a long mark at every whole number, a medium mark at every half-inch, and a tiny mark every quarter-inch. How many marks will he need to make a standard 6-inch ruler?
7. The circumference of a circular table is 30 feet. If a set of silverware is placed every three feet around the circumference of the table, how many place-settings are there?
8. How many whole numbers less than 100 are multiples of 3 but not multiples of 5?
9. At the pound there are 40 dogs. If 22 have spots and 30 have short hair, what is the fewest number of dogs that can have short hair and spots?
10. How many of the first 729 positive integers are perfect squares, cubes or both?

11. How many of the smallest 1000 positive integers are divisible by 5, 6 and/or 7?

12. An auto dealership sells expensive foreign automobiles. You are loking for a black convertible Porsche. The lot has 50 cars that meet at least one of your criteria: 18 Porsches, 25 black cars, and 16 convertibles. There are 3 black convertibles, 4 black Porsches, and 5 convertible Porsches. How many black convertible Porsches are there?

13. How many straight lines will it take to connect all 5 points shown to each of the other 4?

14. A volleyball league has 16 teams. Every team plays every other team once. How many games are played?

15. The first figure is made of 3 toothpicks, each one unit long. The second figure is made from 9 of the same

toothpicks. How many toothpicks are needed for the 16th figure?

16. Several couples arrive at a dinner party. Each person at the party shakes the hand of every other person, not including his or her spouse. If there were a total of 112 handshakes, how many couples attended the party?

17. Find the sum of the first 1000 positive odd integers.

18. The sum of the first three terms of an arithmetic sequence is 20 and the sum of the next three terms is 25. What is the first term in the sequence?

19. What is the positive difference between the fifth term of the geometric sequence that begins $16, 24, ...$ and the fifth term of the arithmetic sequence that begins $16, 24, ...$?

20. How many geometric sequences of two or more positive integers begin with 1 and end with 1024?

21. Find the perimeter of the smallest triangle whose integer side lengths form a geometric sequence whose common ratio is not equal to 1.

22. Simplify $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + ... + \frac{1}{2^{10}}$.

23. Find the sum $\frac{2}{5} + \frac{4}{25} + \frac{4}{125} + \frac{16}{625} + ...$