AMC 10/12 Geometry 4, Dr Liu Springlight

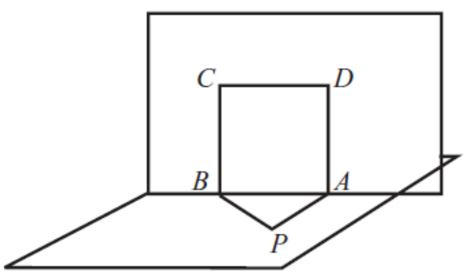
Result 1 Euler's Formula: If a polyhedron with a solid interior has F faces, E edges, and V vertices, then

1. $$F + V - E = 2.$$

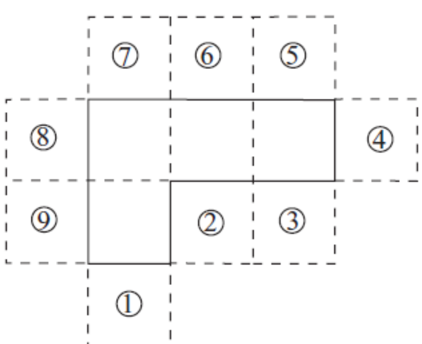
2. A soccer ball is made by stitching together 20 regular hexagons and 12 regular pentagons, each with a side length of 1 inch. What is the combined length of the seams required to stitch together a soccer ball?

3. A snub cube has 6 square faces and 32 triangular faces. What is the total number of vertices and edges on a snub cube?

4. Triangle PAB and square ABCD lie in perpendicular planes. Suppose that $PA = 3$, $PB = 4$, and $AB = 5$. What is PD?

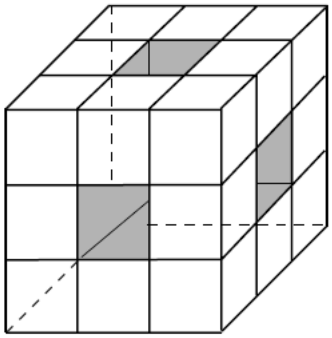


5. The polygon enclosed by the solid lines in the figure consists of 4 congruent squares joined edge-to-edge. One more congruent square is attached to an edge at one of the nine positions indicated. How many of the nine resulting polygons can be folded to form a cube with one face missing?

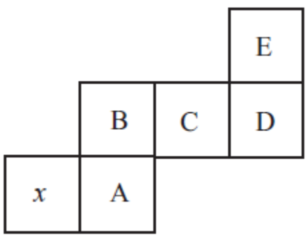


6. A 9×9×9 cube is composed of twenty-seven 3×3×3 cubes. The big cube is "tunneled" as follows: First, the 3×3×3 cubes which make up the center of each face are removed as well as the interior center $3 \times 3 \times 3$ cube, as shown. Second, each of the twenty remaining $3 \times 3 \times 3$ cubes is diminished in the same way. That is, the unit cubes in the

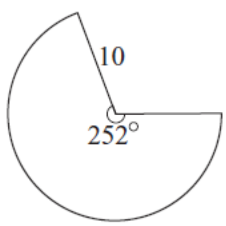
center of each face as well as each interior center cube are removed. What is the surface area of the final figure?

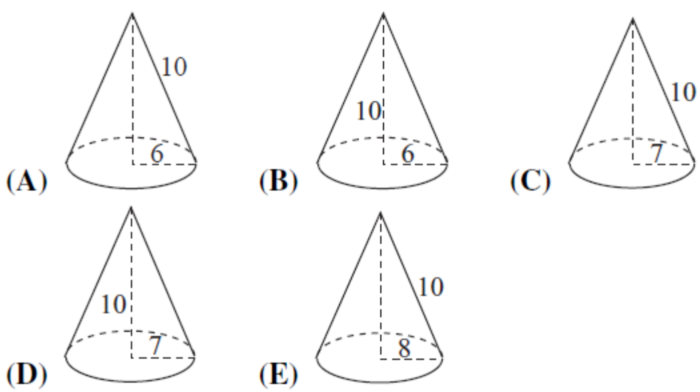


7. The figure shown can be folded into the shape of a cube. In the resulting cube, which of the lettered faces is opposite the face marked x?



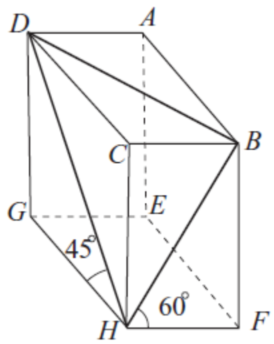
8. Which of the cones below can be formed from a 252 sector of a circle of radius 10 by aligning the two straight sides?



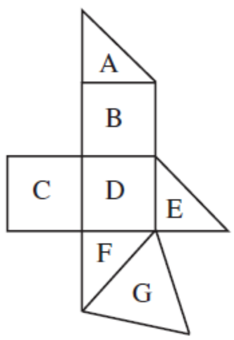


9. Three cubes having volumes 1, 8, and 27 are glued together at their faces. What is the smallest possible surface area that the resulting polyhedron can have?

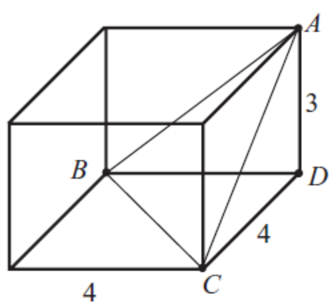
10. In the rectangular solid shown, we have $\angle DHG = 45$ and $\angle FHB = 60$. What is the cosine of $\angle BHD$?



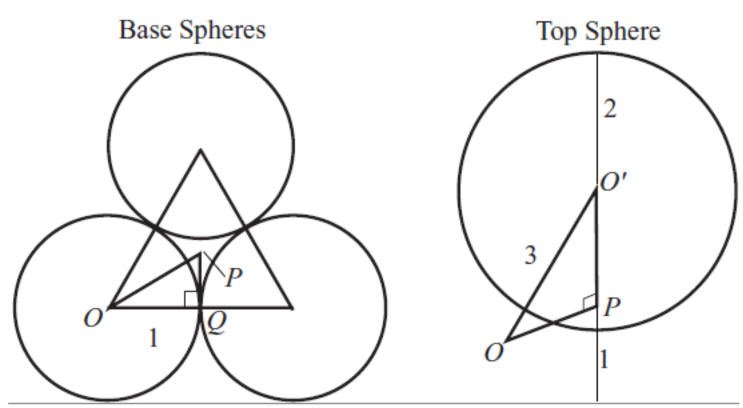
11. In the figure, A, E, and F are isosceles right triangles; B, C, and D are squares with sides of length 1, and G is an equilateral triangle. The figure can be folded along the edges of these polygons to form a polyhedron. What is the volume of the polyhedron?



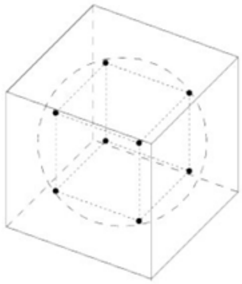
12. On a $4 \times 4 \times 3$ rectangular parallelepiped, vertices A, B, and C are adjacent to vertex D. Consider the plane containing the points A, B, and C. What is the perpendicular distance from D to this plane?



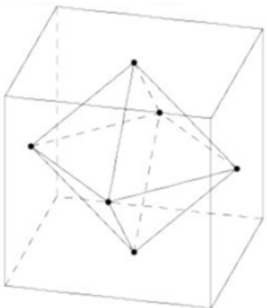
13. Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?



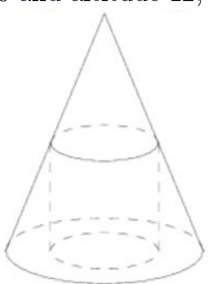
14. A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?



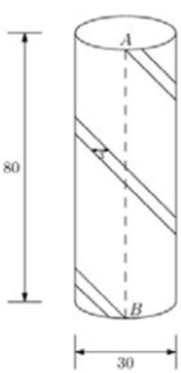
15. Centers of adjacent faces of a unit cube are joined to form a regular octahedron. What is the volume of this octahedron?



16. A right circular cylinder with its diameter equal to its height is inscribed in a right circular cone. The cone has diameter 10 and altitude 12, and the axes of the cylinder and cone coincide. Find the radius of the cylinder.

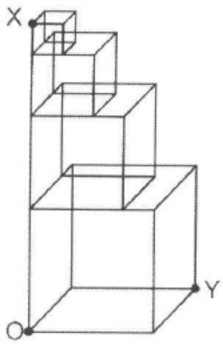


17. A white cylindrical silo has a diameter of 30 feet and a height of 80 feet. A red stripe with a horizontal width of 3 feet is painted on the silo, as shown, making two complete revolutions around it. What is the area of the stripe in square feet?

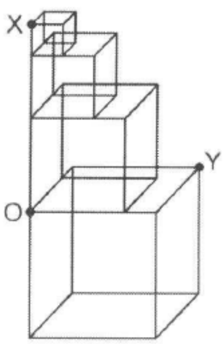


18. (95) Suppose we have a cube with side length 4. In the middle of each face of the cube, cut a 2 by 2 square hole all the way through the cube. What is the volume of the remaining solid after all the holes are cut?

19. (97) Stack cubes with side length 1,2,3,4 as shown. (a) Find the distance from X to Y. (b) Find the length of the portion of $\overline{XY}$ contained in the bottom cube.

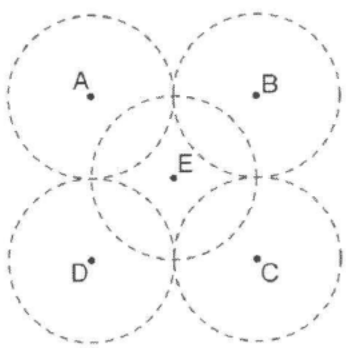


20. (103) Stack cubes with side length 1,2,3,4 as shown. (a) Find the distance from X to Y. (b) Find the length of the portion $\overline{XY}$ not contained inside the cubes.

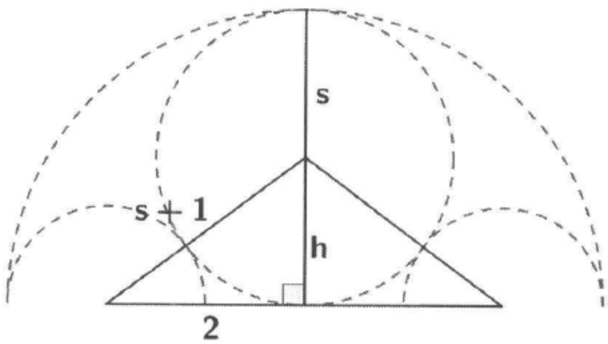


21. (98) Suppose a bored bee lives on a cube with side length 1. For fun he decides to visit every vertex of the cube, each exactly once, starting and ending at the same vertex. It will travel from one vertex to another using a straight line, either by crawling or flying. Give an example of a path that uses the maximum distance and find this distance.

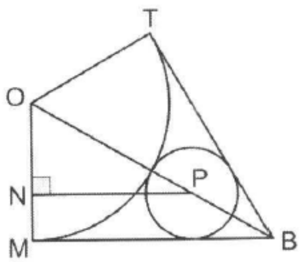
22. (98) Four identical spheres, each of radius 1, are glued to the ground so that their centers form the vertex of a square with side length 2. (a) A fifth identical sphere rests on the 4 spheres so it is externally tangent to the other spheres. How far does this sphere rests off the ground? (b) Suppose the fifth sphere is not the same size as the other 4 and that it rests 1 unit offer the ground. What is the radius of this fifth sphere?



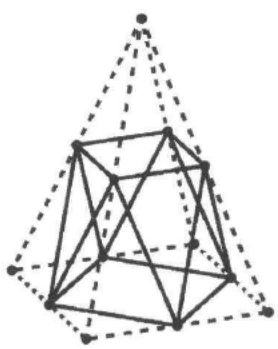
23. (105) Suppose 6 spheres of radius 1 are arranged so that their centers from a regular hexagon with side length 2. All 6 spheres are internally tangent to a larger 7th sphere whose center is the center of the hexagon. Lastly, an 8th sphere is externally tangent to the 6 smaller spheres and internally tangent to the larger sphere. (a) What is the radius of the large (7th) sphere? (b) What is the radius of the 8th sphere?



24. (99) (spheres in a cone) (a) Find the volume of the largest sphere that can fit within a cone of radius 1 and height $\sqrt{3}$. (b) Assume the sphere in (a) is now in place inside the cone. Suppose you want to fit another sphere in the cone that is tangent to the base. Find the radius of the largest such sphere.

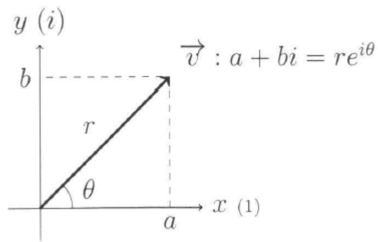


25. (101) Suppose you have a regular square pyramid S-ABCD with height 6 whose square has side length 4. Call the midpoints of the square E, F, G, H (on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ respectively) and the midpoints of $\overline{SA}$, $\overline{SB}$, $\overline{SC}$, $\overline{SD}$ respectively T, U, V, W. Form the polyhedron EFGH-TUVW (with 10 faces). (a) Describe the faces of EFGH-TUVW. How many vertices and edges does the polyhedron have? (b) Find the volume of EFGH-TUVW.



26. (107) You start with a right cone and cut off the top of the cone with a plane parallel to the base. The resulting frustum has 2 circular bases, with radii R and r ($R > r$) and height H. (a) Show that the volume of a frustum is $\frac{\pi H}{3}(R^2 + Rr + r^2)$. (b) Suppose a sphere can be inscribed in a frustum with base radii r, R such that the sphere is tangential to the two bases and the side. Find the radius of such a sphere in terms of r, R.

27. Vectors



argument, which is written as

$\theta = \arg(\overrightarrow{v}) = \arg(a + bi)$ $\quad |\overrightarrow{v}| = r = \sqrt{a^2 + b^2}$

Every vector is corresponding to a complex number. For example, $\overrightarrow{v}$ in the diagram above is corresponding to $a + bi$. This complex number can also be written as $r(\cos\theta + i\sin\theta)$ or $re^{i\theta}$, where $a = r\cos\theta$ and $b = r\sin\theta$.

A vector can start from any point, not necessarily the origin. If $\vec{v}$ starts at $A(x_a, y_a)$ and ends at $B(x_b, y_b)$, then:

$$\vec{v} = \overrightarrow{AB} = (x_b - x_a) + (y_b - y_a)i \tag{7.3}$$

Its magnitude can be computed as:

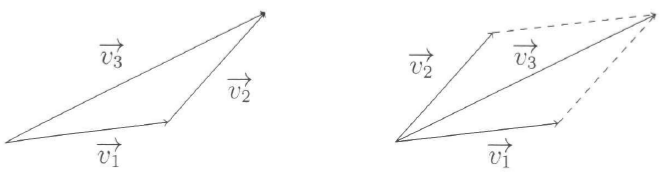
$$|\vec{v}| = |AB| = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}$$

Accordingly, its *argument* equals[2]:

$$arg(\vec{v}) = \arctan \frac{y_b - y_a}{x_b - x_a}$$

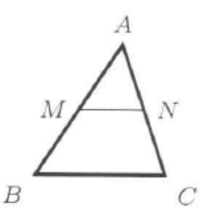
In order to prove two line segments, AB and CD, are parallel, it is sufficient to show $\overrightarrow{AB} = k\ \overrightarrow{CD}$ where k is a scalar. Furthermore, the ratio of their lengths equals $|k|$.

Specifically, if $|k| = 1$, these two segments are not only parallel, but also equal in length.

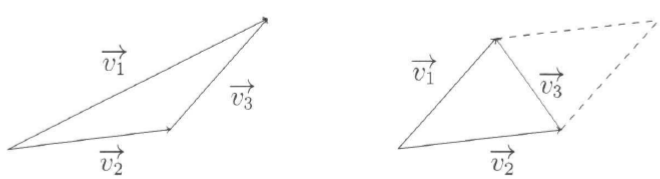


$$\vec{v_3} = \vec{v_1} + \vec{v_2}$$

28. In $\triangle ABC$, show that if M and N are the mid points of $\overline{AB}$ and $\overline{AC}$, respectively, then $MN \| BC$ and $BC = 2MN$.



Vector subtraction can also be visualized using the triangle rule or the parallelogram rule, as shown below:



$$\vec{v_3} = \vec{v_1} - \vec{v_2}$$

While in most cases a vector and its corresponding complex number can be used interchangeably, it is not the case when multiplication operation is involved.

Caution: multiplying two complex numbers is not the same as multiplying two vectors.

Let $\overrightarrow{v_1} = a + bi$ and $\overrightarrow{v_2} = c + di$. Then it is perfectly acceptable to write the following expression:

$$\overrightarrow{v_1} + \overrightarrow{v_2} = (a + bi) + (c + di) = (a + c) + (b + d)i$$

The inner product of two vectors is a scalar. Let $\overrightarrow{v_1} = x_1 + y_1 i$ and $\overrightarrow{v_2} = x_2 + y_2 i$, then

$$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = x_1 x_2 + y_1 y_2 = |\overrightarrow{v_1}||\overrightarrow{v_2}| \cos\theta \qquad (7.4)$$

where θ is the angle between these two vectors.

Theorem 7.1.1

$$\cos\theta = \frac{\overrightarrow{v_1} \cdot \overrightarrow{v_2}}{|\overrightarrow{v_1}||\overrightarrow{v_2}|} = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}} \qquad (7.5)$$

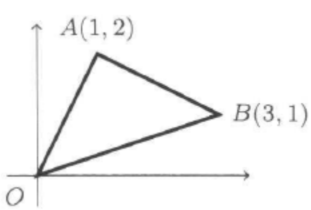
Consider a triangle whose vertices are:

$(0, 0)$, (x_1, y_1) and (x_2, y_2). Accordingly, lengths of its three sides are $\sqrt{x_1^2 + y_1^2}$, $\sqrt{x_2^2 + y_2^2}$, and $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

By the Law of Cosines, we have

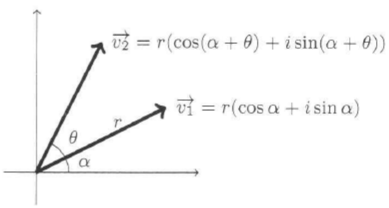
$$\begin{aligned}
&\cos\theta \\
&= \frac{(\sqrt{x_1^2 + y_1^2})^2 + (\sqrt{x_2^2 + y_2^2})^2 - (\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2})^2}{2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}} \\
&= \frac{2x_1 x_2 + 2y_1 y_2}{2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}} \\
&= \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}
\end{aligned}$$

29. Find the angle measure of $\angle AOB$ where O is the origin:



In the diagram below, the vector $\overrightarrow{v_2}$ is obtained by rotating $\overrightarrow{v_1}$ anticlockwise by angle θ.

30.

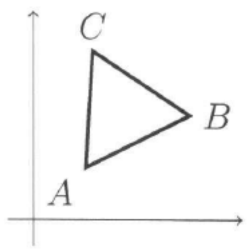


The question is: how to derive $\overrightarrow{v_2}$ using the original vector $\overrightarrow{v_1}$ and angle θ?

Theorem 7.1.2 Vector Rotation

To rotate an vector anticlockwise by angle θ is equivalent to multiplying the vector by $(\cos\theta + i\sin\theta)$.

Let $\triangle ABC$ be equilateral. If the coordinates of A and B are $(1,1)$ and $(3,2)$, respectively, find the coordinates of C.

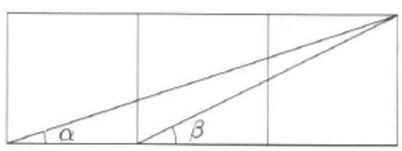


Theorem 7.1.3 Sum of Arguments

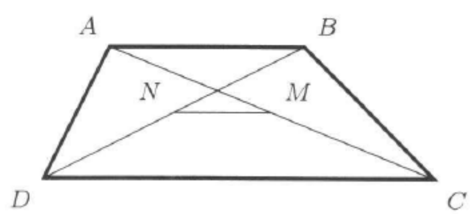
Let z_1 and z_2 be two complex numbers, then

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

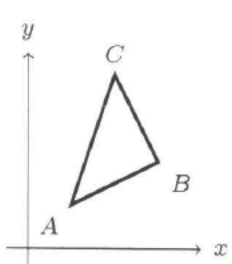
31. Three congruent squares are arranged side by side as shown. What is $\alpha + \beta$?



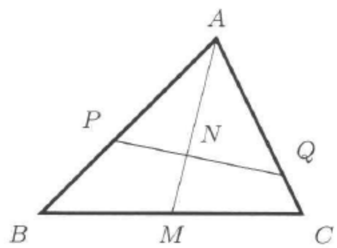
32. Let ABCD be a trapezoid. Points M and N are the mid points of its diagonal AC and BD, respectively. Show that $MN \| AB$ and $MN = \frac{1}{2} \mid AB - CD \mid$.



33. Let $\triangle ABC$ be an isoceles right triangle where $AB = CB$. If the coordinates of A and B are $(1,1), (3,2)$, respectively, find C's coordinates.



34. Let AM be the median of $\triangle ABC$. An arbitrary line intersects AB, AM and AN at points P, N, Q respectively. If $\frac{AB}{AP} = \frac{3}{2}$, $\frac{AC}{AQ} = \frac{4}{3}$, what is $\frac{AM}{AN}$?


A
P
N
Q
B
M
C

AMC10/12 Number Theory 4

1. Let A be the sum of the digits of 5^{10000}, and B be the sum of digits of A, and C be the sum of digits of B. What is C?
2. In triangle ABC, all three sides are integers. Assume that AB = 21, and its perimeter is 54. The area is an integer. What are BC and CA?
3. From the set {1,2,3,...,100}, select k numbers. What is minimum of k guaranteed to have 2 numbers that are not relatively prime?
4. Put a positive or negative sign in front of each of 1,2,3,4,...,2003 and take its sum, then this sum must be
   a. Odd b. Even c. Multiple of 3 d. None of the above?
5. Find all ordered triples (x, y, z) of prime numbers satisfying equation $x(x + y) = z + 120$.
6. Given that a and b are both prime numbers and $p = a^b + b^a$ is also prime. What is p?
7. Given 4 cards with red, yellow, white and blue colors, each card having a digit on it. Mike put the cards in a row in the order red, yellow, white and blue to form a 4-digit number. Then he calculated the difference between this and 10 times the sum of its digits. He found that no matter what the digit was on the white card the result is always 1998. What are the digits on red, yellow and blue cards?
8. In a magic show, the magician asked Nick to think about a 3-digit number $\overline{abc}$; and (1) write down 5 numbers: $\overline{acb}, \overline{bac}, \overline{bca}, \overline{cab}, \overline{cba}$. Then (2) add up these 5 numbers to get N. As soon as Nick said the number N, the magician announced the original number $\overline{abc}$. If N=3194, what was $\overline{abc}$?
9. From the natural numbers 1,2,3,4,...,1000, at most how many can be selected and form a set such that the sum of any three members of the set is a multiple of 18?
10. Find positive integer n that is divisible by both 5 and 49 and has exactly 10 positive divisors?
11. Is it possible to express 99^{99}as the sum of 99 consecutive odd positive integers? If so, how? If not, why not?
12. The 3X3 table below contains 9 prime numbers. Define an operation as adding the same positive integer to the 3 numbers in one row or one column. Is it possible to change all numbers in the table to the same number after several operations? If so, how?

2	3	5
13	11	7
17	19	23

13. Find the sum of all the digits in the numbers 1,2,3,4,...,9999999 (seven 9's)
14. $\frac{m}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2016}$ where gcd(m,n)=1. Show that 2017 | m.
15. One integer n equals the sum of 4 distinct fractions of the form $m/(m + 1)$ (m is a positive integer). Find this integer n, and also find at least one such set of 4 fractions that add up to n.
16. Let n be the smallest multiple of 75 that has exactly 75 factors. Find $n/75$.
17. Find 3 prime numbers whose product is five times their sum.
18. Form 4-digit numbers with the digits 0,1,2,3,4 with no repeating digits within each number. Find the sum of all such 4-digit numbers.
19. A magic square is a square matrix with the property that the sums of the numbers in each row, column and diagonal are the same. The sum is called the magic sum. Is it possible that a 3X3 magic square has a magic sum of 2017?

20. Is it possible to express the fractions $7/332$ and $1949/1999$ in the form $1/m + 1/n$ where m, n are positive integers? If so, what is m and n.

21. Evaluate the sum: $\lfloor \frac{199 \cdot 1}{97} \rfloor + \lfloor \frac{199 \cdot 2}{97} \rfloor + ... + \lfloor \frac{199 \cdot 96}{97} \rfloor$

22. How many even numbers are there among the first 100 Fibonacci numbers?

23. There are 909 numbers on the board 1,2,3,...,909. Repeat the following step: erase any 2 numbers and write their non-negative difference onto the board, until there is only one number left. Is this number odd, even or zero?

24.

$$a \equiv b \ (mod\ m), c \equiv d \ (mod\ m), k \text{ non zero integer} \Rightarrow$$
$$a \pm c \equiv b \pm d \ (mod\ m),$$
$$ac \equiv bd \ (mod\ m),$$
$$\frac{a}{k} \equiv \frac{b}{k} \ (mod\ \frac{m}{\gcd(m,k)})$$
$$a \equiv b \ (mod\ m), k \text{ positive integer} \Rightarrow a^k \equiv b^k \ (mod\ m)$$

If a is an integer, p is a prime number and a is not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$

A frequently used corollary of Fermat's Little Theorem is $a^p \equiv a \pmod{p}$. The restated form is nice because we no longer need to restrict ourselves to integers a not divisible by p.

25.

This theorem is a special case of Euler's Totient Theorem, which states that if a and n are integers, then $a^{\phi(n)} \equiv 1 \pmod{n}$, where $\phi(n)$denotes Euler's totient function. In particular, $\phi(p) = p - 1$ for prime numbers p.

Given a positive integer n, $\varphi(n)$ returns the number of positive integers not exceeding n which are co-prime to n.

$$\varphi(1) = 0, \varphi(2) = 1, \varphi(3) = 2, \varphi(4) = 2, \varphi(5) = 4$$

(correction: a and n need to be co-prime)

Let n's prime factorization be $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \ldots {p_k}^{\alpha_k}$

$$\text{then } \varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_k}\right)$$

(1) If p is prime, then $\varphi(p) = p - 1$

(2) Let p be prime, k a positive integer, then $\varphi(p^k) = p^{k-1}(p - 1)$

(3) Let a and b be co-prime positive integers, then $\varphi(ab) = \varphi(a)\varphi(b)$

26. Let a>b>c . If their remainders are 2,7,9 respectively when divided by 11, find the remainder when $(a+b+c)(a-b)(b-c)$ is divided by 11.

27. Let four integers a,b,c,d satisfy $a + b + c + d = 2017$. Is $a^3 + b^3 + c^3 + d^3$ even or odd?

28. If 17!=$\overline{355687ab8096000}$ where a,b are two missing single digits. Find a and b.

29. Let $a_1, a_2, \ldots, a_{1024}$ be a random arrangement of 1,2,...,1024. Let $b_1, b_2, \ldots, b_{512}$ be a random arrangement of$|a_1 - a_2|, |a_3 - a_4|, \ldots, |a_{1023} - a_{1024}|$. Let $c_1, c_2, \ldots c_{256}$ be a random arrangement of $|b_1 - b_2|, |b_3 - b_4|, \ldots, |b_{255} - b_{256}|$. Repeat this process until a single number N is obtained. Is N even or odd?

30. Find all positive integers n so that $2^n + 1$ is divisible by 3.

31. Find the remainder when $3^{2017} + 4^{2017}$ is divided by 5.

32. Compute 3^{17} (mod 100)

33. Compute 9^{50} (mod 100)

34. What is $\varphi(9)$?

35. What are the last 2 digits of 3^{2017}?

36. Let x and y be two integers. y is the inverse of x modulo m iff $xy \equiv 1\ (mod\ m)$ which can also be written as $y \equiv x^{-1}\ (mod\ m)\ or\ y \equiv \frac{1}{x}\ (mod\ m)$. Inverse of x modulo m exists iff x and m are co-prime, i.e. gcd(x,m) = 1.

37. Compute the inverse of 5 modulo 3 by trying 1,2,3,...
Compute by Euler's theorem, $a^{-1} \equiv a^{\varphi(m)-1}(mod$ m)

38. (Wilson's theorem) An integr p>1 is a prime number iff $(p-1)! \equiv -1(mod\ p)$
Find the remainder when 2015! is divided by 2017.

39. Find the units digit of 7^{2017}

40. Using the fact that $2^{12} = 4096 \equiv 96 \equiv -4 = -2^2 (mod\, 100)$, compute $8^{88}(mod 100)$

41. Compute 7^{2017}(mod 100)

42. Find the last 3 digits of 7^{10000} and 7^{9999}

43. Find the remainder when 2014! is divided by 2017.

44. Given that $9^{50} \equiv 1(mod\, 100)$,find the last 2 digits of $9^1 + 9^2 + 9^3 + ... + 9^{2000}$

45. Find the remainder when 7×8×9×15×16×17×23×24×25×43 is divided by 11.